**Thm:-** **Comparison test!**

Let $x_n$ and $y_n$ be real sequences and suppose that for some $K \in \mathbb{N}$ we have

$$0 \le x_n \le y_n \quad \text{for } n \ge K$$

then

(a) The convergence of $\Sigma y_n$ implies the convergence of $\Sigma x_n$.

(b) Divergence of $\Sigma x_n$ implies divergence of $\Sigma y_n$.

**Examples:-**

(1.) $\Sigma \frac{1}{n^2+1}$ converges.

As $n^2+1 > n^2 \quad \forall n \in \mathbb{N}$.

$\Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2}$

As $\Sigma 1/n^2$ converges

$\Rightarrow \Sigma \frac{1}{n^2+1}$ converges.

(2.) $\Sigma \frac{1}{\sqrt{n}}$ diverges

As $\sqrt{n} \le n \quad \forall n \in \mathbb{N}$

$\Rightarrow \frac{1}{\sqrt{n}} \ge \frac{1}{n}$

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As $\sum 1/n$ diverges

$\Rightarrow \sum \frac{1}{\sqrt{n}}$ diverges.

**Thm:- (Limit Comparison test)-**

Let $(x_n)$ and $(y_n)$ are strictly positive sequences and suppose that following limit exists in $\mathbb{R}$.

$$r := \lim \left( \frac{x_n}{y_n} \right)$$

(a) if $r \neq 0$, then $\sum x_n$ converges if and only if $\sum y_n$ converges

(b) If $r = 0$, If $\sum y_n$ converges, then $\sum x_n$ converges.

**Note:-** If $r = 0$, then divergence of $\sum y_n$ does not say anything about convergence or divergence of $\sum x_n$.

**Examples:-**

(1) $\sum \frac{1}{n^2 n+1}$ is convergent.

Let $x_n = \frac{1}{n^2 - n + 1}$ and $y_n = \frac{1}{n^2}$

then $\frac{x_n}{y_n} = \frac{n^2}{n^2 - n + 1} = \frac{1}{1 - (1/n) + (1/n^2)}$

and $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \lim_{n\to\infty} \frac{1}{1-(1/n)+(1/n^2)} = 1.$

And as $\sum y_n$ converges.

By limit Comparison test (a)

$\sum x_n$ ~~$\sum y_n$~~ converges.

② $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent

let $x_n = \frac{1}{\sqrt{n+1}}$ & $y_n = \frac{1}{\sqrt{n}}$

then $\frac{x_n}{y_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1+1/n}}$

$\lim \frac{x_n}{y_n} = 1$

As $\sum y_n$ diverges.

By limit comparison test (a)

$\sum x_n$ diverges.

③ $\sum \frac{1}{n!}$ is convergent

let $x_n = \frac{1}{n!}$ and $y_n = \frac{1}{n^2}$ for $n \geq 4$.

then $\frac{x_n}{y_n} = \frac{n^2}{n!} = \frac{n}{1.2..(n-1)}$

then $\lim \frac{x_n}{y_n} = 0$

As $\sum y_n$ converges.

By limit comparison test (b).

$\sum x_n$ converges.

Absolute Convergence

Def$^n$:- Let $(x_n)$ be a sequence in $\mathbb{R}$. The series $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ converges in $\mathbb{R}$.

A series $\sum x_n$ is said to be conditionally convergent if $\sum x_n$ is convergent but $\sum |x_n|$ does not converges.

Thm:- If a series in $\mathbb{R}$ is absolutely convergent, then it is convergent.

Proof:- Let $\sum x_n$ be a series in $\mathbb{R}$ and it is absolutely convergent.

$\therefore$ $\sum |x_n|$ converges

To show:- $\sum x_n$ also converges.

As $\sum |x_n|$ converges

$\therefore$ By Cauchy Criterion of series for $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$|x_{n+1}| + |x_{n+2}| + \dots + |x_m| < \epsilon \quad \forall \ m > n \geq K \quad \text{①}$$

Let $(S_n)$ be the sequence of partial sums of $\sum x_n$

then

$$|S_m - S_n| = |x_{n+1} + x_{n+2} + \dots + x_m|$$

$$\leq |x_{n+1}| + \dots + |x_m| \quad \text{(By triangle inequality)}$$

$$< \epsilon \quad \forall \ m > n \geq k \quad \text{(from eq ①)}$$

$$\therefore \ |S_m - S_n| < \epsilon \quad \forall \ m > n \geq k.$$

$\Rightarrow$ $(S_n)$ is a Cauchy sequence

$\Rightarrow$ $(S_n)$ converges in $\mathbb{R}$.

$\Rightarrow$ $\sum x_n$ converges in $\mathbb{R}$.

<u>Example</u>:- (This series is called Alternating series)

① $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Let $x_n = \frac{(-1)^{n+1}}{n}$

then $|x_n| = 1/n$

As $\sum 1/n$ diverges $\Rightarrow$ $\sum |x_n|$ diverges.

We now only need to show that $\sum \frac{(-1)^{n+1}}{n}$

converges.

let $(s_n)$ be the sequence of partial sum of $\sum u_n$

so $\cancel{s_1 = 1}$

$\cancel{s_2 = 1 + \frac{1}{2}}$

$\cancel{s_3 = 1 - 1 + \frac{1}{2} - \frac{1}{3}}$

so $s_1 = 1$

$s_2 = 1 - \frac{1}{2}$

$s_3 = 1 - \frac{1}{2} + \frac{1}{3}$

$\vdots$

then consider $(s_{2n})$ subsequence of $(s_n)$,

$s_2 = 1 - \frac{1}{2}$

$s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$\vdots$

$$s_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

then each term in brackets is positive

so $s_{2n}$ is an increasing sequence & $0 < s_{2n}$

consider $(s_{2n+1})$, then

$$s_{2n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1}$$

$$= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \dots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$$

then $s_{2n+1}$ is ~~dec~~ decreasing sequence & $s_{2n+1} < 1$

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Also, $s_{2n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1}$

$\Rightarrow \quad s_{2n+1} = s_{2n} + \frac{1}{2n+1}$

$\therefore \quad s_{2n} < s_{2n+1} \quad \forall n.$

As $0 < s_{2n}$ and $s_{2n+1} < 1$.

We have $0 < s_{2n} < s_{2n+1} < 1$

$\therefore$ $(s_{2n})$ & $(s_{2n+1})$ are bounded sequences

Also $(s_{2n})$ & $(s_{2n+1})$ are monotone sequence.

$\therefore$ By Monotone convergence theorem,
$(s_{2n})$ and $(s_{2n+1})$ both converges.

Also $(s_{2n})$ & $(s_{2n+1})$ converges to same value.

Therefore $(s_n)$ converges.

$\therefore$ $\sum x_n$ converges.

$\Rightarrow \sum \frac{(-1)^{n+1}}{n}$ converges.

But $\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum \frac{1}{n}$ is not converges.

$\therefore$ $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Test for Absolute convergence:-

Limit Comparison test II :- Suppose that $(x_n)$ and $(y_n)$ are nonzero real sequences and suppose that following limit exists in $\mathbb{R}$.

$$r := \lim \left|\frac{x_n}{y_n}\right| \text{ , then}$$

(a) If $r \neq 0$, then $\sum x_n$ is absolutely convergent if and only if $\sum y_n$ is absolutely convergent.

(b) If $r = 0$, then if $\sum y_n$ is absolutely convergent, then $\sum x_n$ is absolutely convergent.

Root test:- Let $(x_n)$ be a sequence in $\mathbb{R}$ and suppose that limit

$$r := \lim |x_n|^{1/n}$$

exists in $\mathbb{R}$. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$. (Test fails if we get $r = 1$).

Examples:-

① $\sum \frac{2^n}{e^n}$

$\rightarrow$ Let $x_n = \frac{2^n}{e^n}$

$$r = \lim |x_n|^{1/n} = \lim \left(\frac{2^n}{e^n}\right)^{1/n} = \lim 2/e = 2/e$$

then $r = 2/e < 1$.

$\therefore \sum \frac{2^n}{e^n}$ converges.

(2) $\sum \frac{n^n}{e^n}$

let $x_n = \left(\frac{n}{e}\right)^n$

$r = \lim |x_n|^{1/n} = \lim \frac{n}{e}$ does not exist.

So root test fails.

(3.) $\sum \frac{n}{2^n}$

let $x_n = \frac{n}{2^n}$

$$r = \lim |x_n|^{1/n} = \lim \left(\frac{n}{2^n}\right)^{1/n}$$

$$= \lim \frac{n^{1/n}}{2} = \frac{1}{2} \quad \left(\text{As } \lim n^{1/n} = 1\right)$$

$\therefore r < 1/2$.

By Root test, $\sum \frac{n}{2^n}$ converges.

<u>Ratio test:-</u> Let $(x_n)$ be a nonzero sequence in $\mathbb{R}$ and suppose that the limit

$$r := \lim \left|\frac{x_{n+1}}{x_n}\right|$$

exists in $\mathbb{R}$. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$. (Test fails if $r = 1$)

Examples

(1) $\sum \frac{n}{2^n}$

Let $x_n = \frac{n}{2^n}$

$$r := \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n+1 \cdot 2^n}{2^{n+1} \cdot n} \right|$$

$$= \lim \left| \frac{1+1/n}{2} \right| = \frac{1}{2}$$

As $r<1$, $\therefore \sum x_n$ converges By Ratio test.

(2) If we get $r=1$, then Ratio test fails, As

Let $x_n = \frac{1}{n}$

then $r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n}{n+1} \right| = 1$

Here $r=1$ and $\sum 1/n$ diverges. (As done earlier)

Now let $x_n = 1/n^2$

then $r = \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n^2}{(n+1)^2} \right| = 1$

Here also $r=1$, but $\sum 1/n^2$ converges.

Thus $r=1$, $\sum x_n$ may converge or diverge.

## Alternating series:-

**Def$^n$**:- A sequence $(x_n)$ of non zero real nos. is said to be alternating if the terms $(-1)^{n+1}x_n$, $n \in \mathbb{N}$, are all positive (or all negative) real numbers. If the sequence $(x_n)$ is alternating, we say that the series $\sum x_n$ is an alternating series.

**Examples.**

(1.) Let $x_n = (1, -1, 1, -1, \ldots)$

then $(-1)^{n+1}x_n = (1, 1, 1, \ldots)$ sequence of positive real nos.

$\therefore$ $(x_n)$ is an alternating sequence.

(2.) Let $x_n = (1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots)$

then $(-1)^{n+1}x_n = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ is a sequence of positive real nos.

$\therefore$ $(x_n)$ is an alternating sequence.

(3.) Let $x_n = (-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \ldots)$

then $(-1)^{n+1}x_n = (-1, \frac{-1}{4}, \frac{-1}{9}, \frac{-1}{16}, \ldots)$ is a seq. of negative real nos. Hence $(x_n)$ is alternating seq.

## Alternating series test (or Leibnitz's test)

If the alternating series

$$\sum (-1)^{n+1} x_n = x_1 - x_2 + x_3 - x_4 + \cdots$$

satisfies

(a) $x_n > 0 \quad \forall n \in \mathbb{N}$

(b) $x_{n+1} \le x_n \quad \forall n$ (i.e. $(x_n)$ is decreasing seq.)

(c) $\lim x_n = 0$

then the series $\sum (-1)^{n+1} x_n$ converges.

**Examples:-**

1. $\sum \frac{(-1)^n}{n}$

Here $x_n = \frac{1}{n}$ and $x_n \geq 0 \ \forall n$.

and as $n+1 > n$

$\Rightarrow \frac{1}{n+1} < \frac{1}{n}$

$\Rightarrow x_{n+1} < x_n \quad \forall n$.

$\Rightarrow (x_n)$ is a decreasing sequence

Also, $\lim\limits_{n \to \infty} x_n = \lim \frac{1}{n} = 0$

$\therefore$ By Leibnitz's test, $\sum (-1)^n n$ converges.

(8.) $\sum \frac{(-1)^{n+1}}{n^2+1}$

Here $x_n = \frac{1}{n^2+1}$ and $x_n \geq 0 \ \forall n$.

as $n^2+1 < (n+1)^2+1 \ \forall n$

$\Rightarrow \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1}$

$\Rightarrow x_n > x_{n+1} \ \forall n$

$\therefore (x_n)$ is a decreasing seq.

Also, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$

$\therefore$ By Leibnitz's test,

$\sum \frac{(-1)^{n+1}}{n^2+1}$ converges.

<u>Note</u>:- Series in example (8) is absolutely convergent.

(9.) $\sum \frac{(-1)^{n+1}.n}{n+1}$

Here $x_n = \frac{n}{n+1}$, $x_n \geq 0 \ \forall n$.

And $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0$

$\therefore$ Alternating series test (Leibnitz's test fails)

But By $n^{th}$ term test, we can say that

$$\sum \frac{(-1)^{n+1} n}{n+1} \text{ diverges.}$$

(4.) $$\sum \frac{(-1)^{n+1} n}{n^2+1}$$

Here $x_n = \frac{n}{n^2+1}$, Clearly $x_n \geq 0 \; \forall \; n$.

and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n} = 0$

$\therefore \lim_{n\to\infty} x_n = 0$

Now we check whether $(x_n)$ is decreasing or not for that we use first ~~dis~~ derivative test.

[Please recall first derivative test to check whether a function is increasing or decreasing]

Let $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{x^2+1 - x(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \forall \; x > 1$$

$\therefore$ The function is decreasing for $x > 1$.

$\therefore \quad f(2) < f(1)$

In general $f(n+1) < f(n)$

$\Rightarrow \quad x_{n+1} < x_n \quad \forall\, n \geq 1$

$\therefore$ By Leibnitz's test,

$\sum \frac{(-1)^{n+1} \cdot n}{n^2+1}$ converges.

<u>Exercise:-</u>

Test the following series for convergence and for absolute converges:

(i) $\sum \frac{(-1)^{n+1}}{n+1}$

(iv) $\sum (-1)^{n+1} \cdot \frac{\ln n}{n}$

(ii) $\sum (-1)^n \frac{n!}{2^n}$

(iii) $\sum \frac{(-1)^{n+1}}{n!}$

Chapter-8 Sequence of functions

let $A \subseteq \mathbb{R}$ and suppose for each $n \in \mathbb{N}$, there is a function $f_n : A \to \mathbb{R}$, we say that the $(f_n)$ is a sequence of functions on $A$ to $\mathbb{R}$.

Then for each $x \in A$, such a sequence give rise to a sequence of real numbers,

$$(f_n(x))..$$

Defn:- let $(f_n)$ be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$.

We say the sequence $(f_n)$ converges to $f$ on $A_0$.

~~if for each $x \in A_0$, and $\epsilon > 0$ there exists a $K \in \mathbb{N}$~~

~~such that~~ $\cancel{|f_n(x) - f(x)| < \epsilon \text{ whenever } n \geq K}$.

if for each $x_0 \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$

i.e. for $\epsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq K.$$

In this case we call $f$ to be the limit of sequence $(f_n)$ on $A_0$. and denote as

$$f = \lim (f_n) \text{ on } A_0$$

or $\quad f(x) = \lim f_n(x) \quad$ for $x \in A_0$

**Note:-** The convergence defined above is also known as pointwise convergence.

**Examples:**

1. Show $\lim x/n = 0$ for $x \in \mathbb{R}$

Let $f_n : \mathbb{R} \to \mathbb{R}$ defined as

$$f_n(x) = \frac{x}{n}.$$

$$\therefore f_1(x) = x \quad , \quad f_2(x) = \frac{x}{2} , \quad f_3(x) = \frac{x}{3} , \dots$$

and let $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = 0.$$

We have to show that $\lim\limits_{n\to\infty} f_n(x) = f(x)$

As

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (x/n) = x \lim 1/n$$

$$= x \cdot 0 = 0 .$$

$$\therefore \lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in \mathbb{R} .$$

$$\therefore \lim_{n\to\infty} (x/n) = 0$$

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(2.) Check convergence of $f_n(x) = x^n$.

→ As $f_n(x) = x^n$, $f_1(x) = x$, $f_2(x) = x^2$, - -

if $x = 1$ then $f_n(1) = 1^n = 1$

$\therefore f_n(1) = 1$

$\therefore$ the sequence $(f_n(1))$ converges to 1.

and for $0 \leq x < 1$, $\lim_{n \to \infty} x^n = 0$

also if $-1 < x \leq 0$, $\lim_{n \to \infty} x^n = 0$.

$\therefore$ for $-1 < x < 1$, $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n = 0$

and if $x > 1$, $\lim_{n \to \infty} x^n = +\infty$

$\therefore f_n(x)$ converges on the set $(-1, 1]$ to the function

$$f(x) := \begin{cases} 0 & , -1 < x < 1 \\ 1 & , x = 1 \end{cases}$$

(3.) Check convergence of $f_n(x) = \frac{x}{x+n}$, $x \in R, x \geq 0$.

→ Here $f_n(x) = \frac{x}{x+n}$

So, $f_1(x) = \frac{x}{x+1}$, $f_2(x) = \frac{x}{x+2}$, - -

Now as $x+n > n \quad \forall n$ as $x \geq 0$

$\Rightarrow \quad \frac{1}{x+n} < \frac{1}{n}$

$\Rightarrow \quad \frac{x}{x+n} < \frac{x}{n} \quad , \forall n, \forall x \geq 0.$

So let $g_n(x) = x/n$,

Then $\quad 0 \leq f_n(x) \leq g_n(x) \quad \forall n \in N, \forall x \geq 0.$

And as $\lim_{n \to \infty} g_n(x) = 0$

We have $\lim_{n \to \infty} f_n(x) = 0$

(By Squeeze theorem).

$\therefore \quad \lim_{n \to \infty} \frac{x}{x+n} = 0 \quad , \quad x \geq 0.$

<u>Exercise</u>:-

<u>Q1.</u> Show $\lim_{n \to \infty} \frac{nx}{1+n^2x^2} = 0$ for all $x \in R$.

<u>Q2.</u> Show $\lim_{n \to \infty} \frac{x^2+nx}{n} = x$ for all $x \in R$.